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A Note on the Condition Numbers of the B-Spline Bases

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In this note, improved lower bounds are derived for the sup norm condition numbers of the B-spline bases. Numerical calculations done earlier by de Boor indicate that the improved lower bounds are also upper bounds.

1. INTRODUCTION

In [1] and [2], de Boor has shown that the L_∞ condition number $\kappa_{k,\infty}$ of the B-spline basis of order k is bounded:

$$(\pi/2)^{k-1}/2 \leqslant \kappa_{k,\infty} \leqslant 2k9^{k-1}, \qquad k \geqslant 2. \tag{1.1}$$

The purpose of this note is to give an elementary argument showing that $\kappa_{k,\infty} \geqslant \binom{2k-3}{k-2}/\binom{k-2}{[(k-2)/2]}$, so that (1.1) can be replaced by

$$\frac{k-1}{k}\, 2^{k-3/2} \leqslant \kappa_{k,\infty} \leqslant 2k9^{k-1}, \qquad k \geqslant 2. \tag{1.2}$$

2. B-SPLINES AND CONDITION NUMBERS

Suppose N and k are positive integers and let $\mathbf{t} = (t_1, t_2, \ldots, t_{N+k})$ be real numbers such that

$$t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{N+k}, \qquad t_{i+k} > t_i, \qquad i = 1, 2, \ldots, N. \tag{2.1}$$

Let $B_i = B_{i,k}$, $i = 1, 2, \ldots, N$, be the normalized B-splines on $\mathbf{t}$; i.e.,

$$B_{i,k}(x) = (t_{i+k} - t_i)(-1)^k\, [t_i, \ldots, t_{i+k}]_y\, (x - y)_+^{k-1}$$

where $[t_i, \ldots, t_{i+k}]_y$ means the divided difference with respect to y at the points

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$t_i, \ldots, t_{i+k}$. By (2.1), the N B-splines $B_1, \ldots, B_N$ span an N-dimensional linear space of functions of the form

$$s(x, a) = \sum_{j=1}^{N} a_j B_j(x), \qquad x \in \mathbb{R}.$$

The condition number $\kappa_{k,\mathbf{t},\infty}$ of the B-spline basis relative to the partition $\mathbf{t}$ is defined by

$$\kappa_{k,\mathbf{t},\infty} = \sup_{\|a\|_\infty = 1} \| s(\cdot, a)\|_\infty / \inf_{\|a\|_\infty=1} \| s(\cdot, a)\|_\infty . \tag{2.2}$$

Now the condition number of the B-spline basis of order k is defined by

$$\kappa_{k,\infty} = \sup_{N \geqslant 1} \sup_{\mathbf{t}} \kappa_{k,t,\infty} , \tag{2.3}$$

where the supremum is taken over all partitions satisfying (2.1).

Note that if $s(\cdot, a)$ is any spline of unit norm on a particular partition $\mathbf{t}$ then one obtains a lower bound for $\kappa_{k,\infty}$ by computing the coefficient of s with largest absolute value. Indeed, since

$$\sup_{\|a\|_\infty=1} \| s(\cdot, a)\|_\infty = 1,$$

(2.2) can be written

$$\kappa_{k,\mathbf{t},\infty} = \sup_{a \in \mathbb{R}^N} \| a \|_\infty / \| s(\cdot, a)\|_\infty . \tag{2.4}$$

Take now as the particular partition

$$t_i = -1, \qquad t_{i+k} = 1, \qquad i = 1, \ldots, k. \tag{2.5}$$

Thus $N = k$ and, with no interior knots, the splines are just ordinary polynomials of degree less than k on $[-1, 1]$. From the continuity and normalization properties of the B-splines it is easily seen that

$$B_{ik}(x) = 2^{1-k} \binom{k-1}{i-1} (1-x)^{k-i} (1+x)^{i-1}, \qquad i = 1, \ldots, k; \tag{2.6}$$

i.e., the B-splines are the polynomials used in the definition of the Bernstein polynomials. Consider now the Chebyshev polynomial $T_{k-1}(x) = \cos((k-1) \operatorname{arc\,cos} x)$. Recall Rodrigues' formula for T_{k-1},

$$T_{k-1}(x) = (-1)^{k-1} (1-x^2)^{1/2} \frac{d^{k-1}}{dx^{k-1}} \{(1-x^2)^{k-3/2}\} / (1 \cdot 3 \cdot 5 \cdots (2k-3)).$$

Differentiating the product $(1-x)^{k-3/2}(1+x)^{k-3/2}$, one finds

$$T_{k-1}(x) = (-1)^{k-1} B_{1,k}(x) + \sum_{i=2}^{k} (-1)^{k-i} \binom{2k-3}{2i-3} \Big/ \binom{k-2}{i-2} B_{i,k}(x) \quad (2.7)$$

(see also [3, p. 68]). Since

$$\binom{2k-3}{2i-3} \Big/ \binom{k-2}{i-2} = \frac{(2k-3)(2k-5)\cdots(2k+1-2i)}{1\cdot 3\cdot 5\cdots(2i-3)},$$

the largest coefficient is the middle one, given by

$$d_k = \binom{2k-3}{k-2} \Big/ \binom{k-2}{[(k-2)/2]}. \quad (2.8)$$

Furthermore $\| T_{k-1} \|_\infty = 1$ and the first part of the following theorem is proved:

THEOREM. *Suppose* $\kappa_{k,\infty}$ *is given by* (2.3) *and* d_k *by* (2.8). *Then* $\kappa_{k,\infty} \geqslant d_k$. *Moreover*

$$\frac{k-1}{k} 2^{k-3/2} \leqslant d_k \leqslant \frac{k}{k-1} 2^{k-3/2}, \qquad k \geqslant 2. \quad (2.9)$$

Completion of Proof. We prove (2.9). For this, recall Wallis' inequality

$$2^{2n} \Big/ \left(\pi\left(n+\frac{1}{2}\right)\right)^{1/2} \leqslant \binom{2n}{n} \leqslant 2^{2n}/(\pi n)^{1/2}, \qquad n \geqslant 1.$$

Now

$$2^{2n-1} \Big/ \left(\pi\left(n+\frac{1}{2}\right)\right)^{1/2} \leqslant \binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n} \leqslant 2^{2n-1}/(\pi n)^{1/2}.$$

Hence for k even

$$\left(\frac{k-2}{k-\frac{1}{2}}\right)^{1/2} 2^{k-3/2} \leqslant d_k \leqslant 2^{k-3/2},$$

and for k odd

$$\left(\frac{k-1}{k-\frac{1}{2}}\right)^{1/2} 2^{k-3/2} \leqslant d_k \leqslant \left(\frac{k}{k-1}\right)^{1/2} 2^{k-3/2}.$$

Since $((k-2)/(k-1/2))^{1/2} \geqslant (k-1)/k$ for $k \geqslant 4$ and $d_2 = 1$, $d_3 = 3$, we obtain (2.9). ∎

From computations carried out by de Boor of the numbers

$$D_{k,\infty} = \sup_{N \geqslant 1} \sup_{\mathbf{t}} D_{k,\mathbf{t},\infty}$$

with

$$D_{k,\mathbf{t},\infty} = \max_{1 \leqslant i \leqslant N} 1/\mathrm{dist}_{\infty,(t_{i+1},t_{i+k-1})} (B_i, \mathrm{span}(B_j)_{j \neq i}),$$

it appears that

$$D_{k,\infty} = D_{k,\mathbf{t},\infty}$$

with $\mathbf{t}$ given by (2.5). As shown by de Boor, one has

$$\kappa_{k,\mathbf{t},\infty} = \max_i 1/\mathrm{dist}_\infty(B_i, \mathrm{span}(B_j)_{j \neq i}).$$

Thus $\kappa_{k,\mathbf{t},\infty} \leqslant D_{k,\mathbf{t},\infty}$, and, by the localness, $\kappa_{k,\mathbf{t},\infty} = D_{k,\mathbf{t},\infty}$ for the partition (2.5). Therefore it is not surprising to find that the first few d_k given by

k	2	3	4	5	6	7	8	9	10	11
d_k	1	3	5	$11\frac{2}{3}$	21	$46\frac{1}{5}$	$85\frac{4}{5}$	$183\frac{6}{7}$	$347\frac{2}{7}$	$733\frac{10}{63}$

k	12	13	14	15
d_k	$1399\frac{2}{3}$	$2926\frac{19}{33}$	$5628\frac{1}{33}$	$11{,}688\frac{141}{143}$

agree with the numbers $D_{k,\infty}$ on p. 142 of [2].

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